Feynman-path analysis of Hardy's paradox: measurements and the uncertainty principle

D. Sokolovski[a], I. Puerto Giménez[b], R. Sala Mayato[b] [1]

[a]*School of Mathematics and Physics,*
Queen's University of Belfast, Belfats, BT7 1NN, United Kingdom
[b]*Departamento de Física Fundamental II,*
Universidad de La Laguna, La Laguna, 38204, S/C de Tenerife, Spain

Abstract

Hardy's paradox is analysed within Feynman's formulation of quantum mechanics. A transition amplitude is represented as a sum over virtual paths which different intermediate measurements convert into different sets of real pathways. Contradictory statements emerge when applying to the same statistical ensemble. The "strange" weak values result is also investigated in this context.

PACS: 03.65.Ca, 03.65.Ta

1. Introduction

The Hardy's paradox first introduced in [1] continues to attract attention in the literature [2, 3, 4, 5, 6, 7, 8]. The paradox consists in that, in a two-particle interferometer set up, detection of particle(s) in different arms yields results obviously incompatible with each other. One way to resolve the paradox is by noting that the conflicting results refer not to the same but to different physical situations thereby avoiding counterfactual reasoning - in the words of Ref. [4] "talking about the values of non-measured attributes". The legitimacy of counterfactual statements was explored in [3, 4] within the framework of formal logic. A different resolution involving inaccurate or weak quantum measurements was proposed in [5]. The purpose of this paper is to analyse the Hardy's paradox, counterfactual statements and the weak measurements in terms of virtual (Feynman) paths and the uncertainty principle [9, 10]. In Feynman's quantum mechanics a transition probability amplitude is found by adding amplitudes for all interfering paths which, together, form a single indivisible pathway connecting the initial and final states of the system. The same paths can be either interfering or exclusive alternatives, depending on whether or not the system interacts with other (e.g. meter) degree's of freedom. In addition, Feynman's uncertainty principle states that any determination of the path taken must destroy the interference between the alternatives [9].

The simplest illustration of the dangers of counterfactual reasoning is Young's double-slit experiment. One observes the probabilities with which electron starting in its initial state (source) reaches a variety

---

[1]E-mail address: dsokolovski@qub.ac.uk (D. Sokolovski); rsala@ull.es (R. Sala Mayato)

of final states (points on the screen) via single pathway comprising both slits number 1 and 2 so that an interference pattern is produced. An accurate observation of the slit chosen by an electron produces a system in which each final state can be reached via two real pathways (one through each slit) travelled with probabilities P_1 and P_2. It is a *different* system: the interference pattern is destroyed, and the probabilities to arrive in the final state do not agree with the unobserved ones. Answering the "which way?" question by attributing the probabilities P_1 and P_2 to unobserved system would constitute a counterfactual statement which is, obviously, wrong. Below we will show that the Hardy's set up is equivalent to a three-slit Young experiment, and that, in a similar way, the contradicting statements at the centre of the "paradox" refer to different sets of real pathways produced by different intermediate measurements and not to a single statistical ensemble.

The authors of [5] have come to the defence of counterfactual reasoning suggesting to resolve the paradox with the help of weak measurements [11, 12, 13, 14] performed by a meter whose interaction with the system is so weak, that the interference between different paths is not destroyed. There are two distinct issues associated with the weak measurements. The answer to the first (and easier) question of whether they can be performed in practice is yes [6, 7, 8, 15]. The second (and, to our knowledge not yet fully answered) question concerns the interpretation of the weak results. Weak measurements are often seen "as an extension to the standard von Neumann model of measurements" [8] which allows one to refer to some of the outcomes as "strange and surprising" [5]. The problem is captured by the following simple example: consider applying a weak measurement to determine the slit number in the above double-slit experiment. For a particle arriving near a minimum of the interference pattern one can obtain, say, a number 10 [16] (a similar unusual value has been obtained in the optical realisation of the experiment in [15]). This is not a valid (hence "surprising") slit number as there are only two holes drilled in the screen [17]. Possible interpretations are that (i) the weak measurement reveals some new information about how the particle traverses the screen or, (ii) that in the weak limit the meter is not working properly. Here we will follow Ref. [16] in adopting the second view and turn to the uncertainty principle for an explanation. If the measured result is wrong, what is the correct answer to the "which slit?" question? The uncertainty principle tells us that for an unobserved electron the paths through the first and the second holes form (and this is the only paradox of quantum mechanics [10]) a single indivisible pathway so that the correct answer does not exist. Thus a weak meter must either fail or the uncertainty principle would be proven wrong. The meter does fail by producing an answer which appears to have nothing to do with the original question. Mathematically, a weak value is an improper average obtained with an alternating non-probabilistic amplitude distribution [16] and as such is not tied to its support (in this case, slit numbers 1 and 2), and serves mostly to demonstrate that the particle cannot be seen as passing through a slit with any particular probability. We will argue further that the same can be said about the "resolution" of the Hardy's paradox, which just as the above example relies on the surprising value -1 of the pair occupation number.

The rest of the paper is organised as follows: in Section 2 we briefly describe measurements as a way of converting interfering virtual paths into exclusive real ones. In Section 3 we discuss the three-box case of Aharonov *et al* [14] and its relation to the uncertainty principle. In Section 4 we identify the virtual paths for the Hardy's set up. In Section 5 we analyse real pathways produced from these by different measurements. In Section 6 we consider differences between measurements with and without post-selection. In Section 7 we discuss weak measurements for the Hardy's scheme and show that a different choice of the final state can lead to extremely large "anomalous" weak occupation numbers. Secttion 8 contains our conclusions.

2. Quantum measurements and virtual paths

Consider a quantum system with a zero Hamiltonian $\hat{H} = 0$ in a N-dimesional Hilbert space. The system is prepared (pre-selected) in some state $|i\rangle$ and, at a later time T, observed (post-selected) in a final state $|f\rangle$. It is convenient to choose an othonormal basis $\{|n\rangle\}$, $\quad n = 1, 2..N$ corresponding to the "position" operator [18] $\hat{n} \equiv \sum_{n=1}^{N} |n\rangle n \langle n|$. In general, the transition amplitude can be written as a sum over all virtual paths $n(t)$ which take the values $1, 2...N$ at any given time t [19, 20]. Since $\hat{H} = 0$, there are only N constant paths $n(t) = 1, 2...N$ and the path decomposition of the transition amplitudes takes a simple form

$$\langle f|i\rangle = \sum_{n=1}^{N} \langle f|n\rangle \langle n|i\rangle \equiv \sum_{\{n\}} \Phi\{n\}, \tag{1}$$

where $\Phi\{n\} = \langle f|n\rangle \langle n|i\rangle$ is the amplitude for the n-th path. With these notations the problem becomes equivalent to a N-slit Young experiment with N discrete final destinations (positions on the screen) given by the chosen final state $|f\rangle$ and any $N-1$ orthonormal states spanning the Hilbert subspace orthogonal to $|f\rangle$. The virtual paths provide a convenient way to describe an intermediate von Neumann measurement at $0 < t < T$ of any operator $F(\hat{n})$ which commutes with the position $\hat{n}$,

$$F(\hat{n}) \equiv \sum_{n=1}^{N} |n\rangle F(n) \langle n|, \tag{2}$$

where $F(n)$ is an arbitrary function. For an accurate measurement the initial pointer position f is known exactly, its initial state is the delta-function $\delta(f)$ and the probability amplitude $\Phi(f)$ to register a meter reading f is given by

$$\Phi(f) = \sum_{\{n\}} \delta(f - F(n)) \Phi\{n\}. \tag{3}$$

It is readily seen that if $F(\hat{n})$ has $M < N$ distinct eigenvalues $F_1, F_2,... F_M$ with the multiplicities $m_1, m_2,...m_M$, respectively, N virtual paths are divided into M exclusive classes. Each such class can be seen to form a real indivisible pathway to which one can assign a probability $P_{\{m\}}^{f \leftarrow i}$, with which a reading $f = F_m$ would occur if the measurement is performed.

The transition probabilities are, in general, altered by the measurement,

$$\tilde{P}^{f \leftarrow i} \equiv \sum_{j=1}^{M} |\sum_{n=1}^{N} \delta(F(n) - F_j) \Phi\{n\}|^2 \neq |\langle f|i\rangle|^2. \tag{4}$$

Note that if all eigenvalues of $F(\hat{n})$ are the same, no new real pathways are produced, nothing is measured and no perturbation is incurred. Note also that Eq. (4) is just an example of the Feynman's rule for assigning probabilities [10] which prescribes adding amplitudes for the interfering, and probabilities for the exclusive alternatives.

Another parameter which affects both the obtained information and the incurred perturbation is the accuracy of the measurement Δf. If the meter's sharp initial state is replaced with a distributed one, $\delta(f) \rightarrow G(f)$, where $G(f)$ can be chosen a Gaussian with a width Δf, the amplitude to obtain a reading f is given by

$$\Psi(f) = \int df' G(f - f') \Phi(f') = \sum_{n} G(f - F(n)) \Phi\{n\}. \tag{5}$$

In general, an inaccurate measurement produces a continuum of real pathways labelled by the variable f. The same path $\{n\}$ contributes to many pathways, its contribution to $\Psi(f)$ being $G(f - F(n))\Phi\{n\}$. The perturbation decreases with the increase of the uncertainty Δf and becomes negligible for $\Delta f >> \delta f$ where δf is the difference between the largest and the smallest eigenvalue of $F(\hat{n})$. Application of the meter under these conditions leads to so-called weak measurements first proposed in [11, 12, 13] and recently discussed in [16]. We will return to weak measurements in Section 7.

3. The uncertainty principle and the three-box example

The uncertainty principle states that [9]: "Any determination of the alternative taken by a process capable of following more that one alternative destroys the interference between alternatives". An illustration of what the principle may mean in practical circumstances is provided by the three-box case by Aharonov *et al* [13, 14, 21, 22, 23]. Consider a three-state system with a zero Hamiltonian pre- and post-selected in states such that the amplitudes in (3) are ($\beta < 1$ is real)

$$\Phi\{1\} = \beta, \quad \Phi\{2\} = -\beta \quad and \quad \Phi\{3\} = -\beta \tag{6}$$

so that for an unobserved system we have

$$\langle f|i\rangle = \sum_{\{n\}} \Phi\{n\} = \beta. \tag{7}$$

One sees that there is a cancellation between the paths but cannot decide whether it is paths $\{1\}$ and $\{2\}$ or $\{1\}$ and $\{3\}$ that make each other redundant. All three paths must therefore be treated as a single indivisile pathway, denoted $\{1+2+3\}$, travelled with the probability $P_{\{1+2+3\}}^{f\leftarrow i} = |\Phi\{1\} + \Phi\{2\} + \Phi\{3\}|^2 = \beta^2$.

If one decides to accurately measure the projector on $|2\rangle$, whose matrix form is

$$\hat{P}_2 \equiv diag(0,1,0), \tag{8}$$

in order to see that the particle does indeed follow the path $\{2\}$, the meter will read 1 confirming the assumption and the particle will be detected in $|f\rangle$ with the same probability $P^{f\leftarrow i} = \beta^2$. Perhaps surprisingly, measurement of the projector

$$\hat{P}_3 \equiv diag(0,0,1) \tag{9}$$

will confirm that particle always travels along the path $\{3\}$. Assuming that these results refer to the same system (ensemble) prompts a somewhat paradoxical conclusion that the particle is in several places simultaneously [13, 14]. This "paradox" disappears [23] once one notices that measurement of $\hat{P}_2$ creates a new network in which a final state can be reached via two real pathways, $\{2\}$, and a coherent superposition of the two remaing ones, $\{1+3\}$. In a similar way, measureament of $\hat{P}_3$ fabricates yet another different statistical ensemble with two real pathways, $\{3\}$ and $\{1+2\}$. What is true for one ensemble is not true for the other even though both are produced by observations made on the same system. Equally, neither measurement reveals what "actually" happens in the unobserved system where the "which way?" information is lost, in accordance with the uncertainty principle, to quantum interference.

4. Virtual paths in the Hardy's setup

The Hardy's set up shown in Fig. 1 has been discussed by many authors [1, 2, 5, 6, 7] and a brief description will suffice here. An electron $(e-)$ and a positron $(e+)$ are injected into their respective Mach-Zehnder interferometers, each equipped with two detectors, labelled $C-$, $D-$, $C+$ and $D+$. So long as two interferometers are independent, the outcome of the experiment consists in two detectors clicking in coincidence. There are, therefore, four possible final outcomes. Each particle has a choice of one of the two arms of the corresponding interferometer, so that there are four virtual paths to reach each outcome. The system is, therefore, equivalent to the four-slit Young diffraction experiment, with a minor distinction that there are only four discrete final states (positions on the screen). Hardy changes this arrangement by allowing two interferometer arms, labelled $2-$ and $2+$ overlap so that if both particles are injected simultaneously and choose to travel along the overlapping arms, annihilation follows with certainty. Further, each interferometer is tuned in such a way that neither $D-$ nor $D+$ click if only an electron or a positron is injected. Now the outcome of the experiment consist in two of the four detectors clicking in coincidence or in a photon, γ, produced in annihilation. There are, therefore, five possible final states. Let the orthogonal state $|1-\rangle(|2+\rangle)$ and $|2-\rangle(|1+\rangle)$ correspond to the electron (positron) travelling via non-overlapping (overlapping) arms and vice versa of the corresponding interferometer, respectively. Similarly, the states $|1-\rangle|1+\rangle$ and $|2-\rangle|2+\rangle$ correspond to both particle travelling via non-overlapping and overlapping arms. The four paths can be labelled as follows:

$$
\begin{array}{lll}
\{1\} & via & |1-\rangle|1+\rangle \\
\{2\} & via & |1-\rangle|2+\rangle \\
\{3\} & via & |2-\rangle|1+\rangle \\
\{4\} & via & |2-\rangle|2+\rangle.
\end{array}
\tag{10}
$$

Just past the point P in Fig.1, the state of the electron-positron pair is

$$
|i\rangle = (|1+\rangle|1-\rangle + |1+\rangle|2-\rangle + |2+\rangle|1-\rangle + |\gamma\rangle)/2,
\tag{11}
$$

where $|\gamma\rangle$ is the state of the photon produced in annihilation. Of the five final states, the four corresponding to electron and positron arriving at the detectors $(D-,D+)$, $(C-,D+)$, $(D-,C+)$ and $(C-,C+)$, respectively, are given by

$$
|f\rangle = (|1-\rangle - |2-\rangle)(|1+\rangle - |2+\rangle)/2
\tag{12}
$$

$$
|g\rangle = (|1-\rangle + |2-\rangle)(|1+\rangle - |2+\rangle)/2
\tag{13}
$$

$$
|h\rangle = (|1-\rangle - |2-\rangle)(|1+\rangle + |2+\rangle)/2
\tag{14}
$$

$$
|j\rangle = (|1-\rangle + |2-\rangle)(|1+\rangle + |2+\rangle)/2.
\tag{15}
$$

Figure 2 shows the network of paths connecting the initial and final states of the Hardy's system. There is only one path which connects the initial state with $|\gamma\rangle$, while each of the final states (12-15) can be reached

via each of the pathways $\{1\}$, $\{2\}$ and $\{3\}$. For example, the probability amplitude to reach a final state $|z\rangle$ via pathway $\{1\}$ is given by

$$\Psi_{\{1\}}^{z\leftarrow i} = \langle z|1-\rangle|1+\rangle\langle 1+|\langle 1-|i\rangle, \quad z = f,g,h,j,\gamma. \tag{16}$$

Probability amplitudes for the paths $\{2\}$ and $\{3\}$ can be found in asimilar manner and are given in Table.1. Paths connecting different final states (see Fig.2) are mutually exclusive and cannot interfere. If no additional observations are made on the system, virtual paths ending in the same final state are interfering alternatives and must be treated as a single real pathway. Coherence between such paths can be destroyed, e.g., by an intermediate accurate measurement of an operator. As discussed in Section 2, each such measurement would produce, depending on the multiplicity of the operator's eigenvalues, a number of additional real pathways. In the next Section we will consider these measurements in more detail.

5. Accurate measurements and real pathways in Hardy's set up. Counterfactuals

Following [5] we wish to perform intermediate measurements of "pair occupation" operators which establish whether the electron and the positron are propagating along specified arms of their respective interferometers. In the basis consisting of the states in the r.h.s. of Eq. (10) these projectors take the form

$$\hat{N}(1-|1+) = diag(1,0,0,0) \tag{17}$$

$$\hat{N}(1-|2+) = diag(0,1,0,0) \tag{18}$$

$$\hat{N}(2-|1+) = diag(0,0,1,0) \tag{19}$$

$$\hat{N}(2-|2+) = diag(0,0,0,1). \tag{20}$$

We will also require single particle occupation operators which establish whether the electron (positron) travels along the specified arm, while the position of the other member of the pair remains indeterminate,

$$\hat{N}(1-) = \hat{N}(1-|1+) + \hat{N}(1-|2+) = diag(1,1,0,0) = 1 - \hat{N}(2-) \tag{21}$$

$$\hat{N}(1+) = \hat{N}(1-|1+) + \hat{N}(2-|1+) = diag(1,0,1,0) = 1 - \hat{N}(2+). \tag{22}$$

Consider now accurate measurements of these operators for the system post-selected in $|f\rangle$ (electron and positron are detected in $D-$ and $D+$, respectively). There are only three contributing paths, $\{1\}$, $\{2\}$ and $\{3\}$ with the amplitudes $1/4$, $-1/4$ and $-1/4$, so that the problem becomes equivalent to the three-box case of Section 3. If $\hat{N}(1-|1+)$ is measured accurately, path $\{1\}$ becomes a real pathway, while the second real pathway, $\{2+3\}$, is formed by interfering virtual paths $\{2\}$ and $\{3\}$ as is shown in Fig. 4. The probabilities for these pathways are found by adding, where appropriate, the corresponding amplitudes in Table 1 and squaring the moduli,

$$P_{\{1\}}^{f\leftarrow i} = |\Phi_{\{1\}}^{f\leftarrow i}|^2 = 1/16, \quad P_{\{2+3\}}^{f\leftarrow i} = |\Phi_{\{2\}}^{f\leftarrow i} + \Phi_{\{3\}}^{f\leftarrow i}|^2 = 1/4. \tag{23}$$

Repeating the calculation for the case $\hat{N}(1-|2+)$ is measured yields

$$P_{\{1\}}^{f\leftarrow i} = |\Phi_{\{2\}}^{f\leftarrow i}|^2 = 1/16, \quad P_{\{1+3\}}^{f\leftarrow i} = |\Phi_{\{1\}}^{f\leftarrow i} + \Phi_{\{3\}}^{f\leftarrow i}|^2 = 0, \tag{24}$$

suggesting that if $D+$ and $D-$ click (which happens with a probability of $1/16$) *(I) the electron and the positron always travel along the non-overlapping and overlapping arms, respectively.* Similarly, for $\hat{N}(2-|1+)$ one finds

$$P_{\{3\}}^{f\leftarrow i} = 1/16, \quad P_{\{1+2\}}^{f\leftarrow i} = 0, \tag{25}$$

and might conclude, in contradiction to the above, that *(II) the electron and the positron always travel along the overlapping and non-overlapping arms, respectively.* For the single particle operators (21-22) real pathways and corresponding probabilities can be constructed in a similar manner and are given in Table 2. Thus, measuring $\hat{N}(1-)$ we find that *(III) the electron always travels along the overlapping arm.* Measuring $\hat{N}(1+)$ reveals that *(IV) the positron always travels along the overlapping arm.* The italicised statements $(I),(II),(III)$ and (IV), if referred to the same system, would imply that "and electron and a positron in some way manage to "be" and "not be" at the same time at the same location" [5]. As in Section 3 the "paradoxical" nature of the Hardy's example is removed once one notices that the above statements refer to different networks of classical (real) pathways produced from the same parent unobserved system. That these networks are indeed different is seen already from the fact that the transition probabilities to arrive in the final states $|g\rangle$, $|h\rangle$ and $|j\rangle$ in Table 2 are different for each choice of the measured quantity. Note that for the final state $|f\rangle$ the transition probability remains unchanged, but only due to the special choice of the system's parameters. Thus, only the statement (I) applies under the condition $\hat{N}(1-|2+)$ is measured, while statements (II), (III) and (IV) refer to unmeasured attributes and should be discarded. The approach to resolving quantum interference "paradoxes" based on avoiding counterfactual reasoning is by no means new [3, 4, 5]. We note however, that the path analysis with its notion of converting interfering virtual paths into exclusive real ones, provides a helpful insight into the argument.

6. "Which way?" probabilities without post-selection. The sum and the product rules

In the above $P_{\{1\}}^{f\leftarrow i} = |\Phi_{\{1\}}^{f\leftarrow i}|^2$ gave the probability for the pair to travel along the non-overlapping arms provided the detectors $D+$ and $D-$ click in coincidence. Alternatively, we may choose to record the frequency with which the route is travelled regardless of which detectors click, or just switch the detectors off completely. Bearing in mind that the paths connecting different final states cannot interfere we find the corresponding probability to be

$$P_{\{1\}}^{all\leftarrow i} = |\Phi_{\{1\}}^{f\leftarrow i}|^2 + |\Phi_{\{1\}}^{g\leftarrow i}|^2 + |\Phi_{\{1\}}^{h\leftarrow i}|^2 + |\Phi_{\{1\}}^{j\leftarrow i}|^2 = \langle i|\hat{N}(1-|1+)|i\rangle. \tag{26}$$

The operator average in the r.h.s. of (26) is the standard [24] expression for the probability of an outcome for a system in the state $|i\rangle$, which we have *derived* using (16), completeness of the states (12)-(15) and the fact that $\hat{N}(1-|1+)^2 = \hat{N}(1-|1+)$. Note that summation over all final outcomes has given $P_{\{1\}}^{all\leftarrow i}$ properties not possessed by the state-to-state probabilities $P_{\{1\}}^{z\leftarrow i}$, $z = f,g,h,j$. For example, while $P_{\{1\}}^{z\leftarrow i}$

clearly varies with the choice of the final state $|z\rangle$, $P_{\{1\}}^{all \leftarrow i}$ remains the same for all choices of orthogonal final sates $|f\rangle$, $|g\rangle$, $|h\rangle$ and $|j\rangle$, and only depends on the inital state $|i\rangle$. Also, from Table 2 one notes that (we apologise for the somewhat awkward sentence about to follow) the state-to-state probability for the electron to travel along the non-overlapping arm if $\hat{N}(1-)$ is measured does not equal the sum of probabilites for the electron and positron to travel along non-overlapping arms if $\hat{N}(1 - |1+)$ is measured, and that for the electron to travel along non-overlapping and the positron along the overlapping arms, respectively, provided we measure $\hat{N}(1 - |2+)$, e.g.,

$$P_{\{1+2\}}^{f \leftarrow i} = 0 \neq P_{\{1\}}^{f \leftarrow i} + P_{\{2\}}^{f \leftarrow i} = 1/8 \,, \tag{27}$$

even though $\hat{N}(1-) = \hat{N}(1 - |1+) + \hat{N}(1 - |2+)$. This should not come as a surprise since, as discussed above, measurements of $\hat{N}(1-)$, $\hat{N}(1 - |1+)$ and $\hat{N}(1 - |2+)$ create three different statistical ensembles and Eq. (27) is nothing more than an indication of this fact. An additional "sum rule" is obtained only if the individual "which way?" probabilities are summed over all final states

$$P_{\{1+2\}}^{all \leftarrow i} \equiv \langle i|\hat{N}(1-)|i\rangle = \langle i|\hat{N}(1 - |1+) + \hat{N}(1 - |2+)|i\rangle = P_{\{1\}}^{all \leftarrow i} + P_{\{2\}}^{all \leftarrow i} \,. \tag{28}$$

A closely related subject is the "failure" of the product rule for post-selected systems [2, 13, 25]. As was shown in the previous Section, a measurement of $\hat{N}(2-)$, conditioned on detectors $D+$ and $D-$ clicking at the same time, shows that the electron always travels along its overlapping arm. Measuring $\hat{N}(2+)$ shows that the same can be said about the positron as well (see Table 2). Table 1 shows that a measurement of $\hat{N}(2 - |2+)$ gives $P_{\{4\}}^{f \leftarrow i} = |\Phi^{f \leftarrow i}|^2 = 0$ and reveals, as it should, that the two particles cannot travel the overlapping paths simultaneously as they would annihilate and never reach the detectors. Morover, from Eqs. (17) to (22) it is clear that $\hat{N}(2 - |2+) = \hat{N}(2-)\hat{N}(2+)$. Again, assuming that all three results refer to the same statistical ensemble leads to a contradiction, as the joint probability of two certain events must also equal one. Hence inapplicability of the product rule gives another evidence that with post-selection a different ensemble is produced with each choice of the measured quantity, even though all three operators commute [23]. It is a simple matter to verify that without post-selection conditions $\langle i|\hat{A}|i\rangle = 1$ and $\langle i|\hat{B}|i\rangle = 1$ would force $\langle i|\hat{A}\hat{B}|i\rangle = 1$ for any two commuting operators $\hat{A}$ and $\hat{B}$.

One of the purposes of this Section is to stress the fundamental nature of the Feynman's rule for assigning probabilities [10] which we have used first for constructing $P_{\{1\}}^{z \leftarrow i}$, $z = f, g, h, j$ and then in the derivation of Eq. (26). On the other hand, someone who chooses the operator average in Eq. (26) as a starting point for defining quantum probabilities might find generalisation to pre- and post-selected ensembles more difficult, and the apparent break down of the sum and the product rules an unexpected property of post-selection.

7. Weak measurements in Hardy's set up

In Ref. [5] the authors argued against discarding counterfactual statements on the ground that they can be - to some extent- verified simultaneously provided the accuracy of the measurements is so low that the system remains essentially unperturbed. This can be achieved by making the meter state $G(f)$ in (5) very broad,

$$G(f) \to \alpha^{-1/4} G(f/\alpha), \quad \alpha \to \infty \tag{29}$$

8

so that the mean meter reading $\langle f \rangle$ takes the form [13, 16] ($G(f) = G(-f)$)

$$\langle f \rangle \equiv \int f |\Psi(f)|^2 df / \int |\Psi(f)|^2 df \approx Re \sum_{n=1}^{N} F(n)\Phi\{n\} / \sum_{n=1}^{N} \Phi\{n\} \equiv \bar{F} , \quad (30)$$

where $\bar{F}$ is the weak value of the operator $F(\hat{n})$. Since there are no *apriori* restrictions on the phases of (in general complex valued) amplitudes $\Phi\{n\}$, $\bar{F}$ is an improper non-probabilistic average [16] with the obvious properties

$$\bar{1} = 1, \quad (31)$$

$$\overline{F_1 + F_2} = \bar{F}_1 + \bar{F}_2, \quad (32)$$

and

$$\bar{F} = F(m) \quad if \quad \Phi\{n\} = \Phi\{m\}\delta_{nm}, \quad (33)$$

i.e., in the absence of interference, when a single pathway connects initial and final states of the system.

In the three-box case of Section 3 one finds $\bar{P}_2 = 1$ for the projector $\hat{P}_2$, just as it would be if an accurate measurement was conducted. Similarly, for the projector $\hat{P}_3$ one has $\bar{P}_3 = 1$, which suggests that, since weak measurements do not perturb, the two expectation values, normally observed in two different experiments, can be obtained simultaneously.

With the $|f\rangle \leftarrow |i\rangle$ transition in the Hardy's set up being equivalent to the three-box case we therefore have:

$$\bar{N}(1 - |2+) = \bar{N}(2 - |1+) = 1. \quad (34)$$

This would have confirmed that two scenarios *(i) electron in the non-overlapping, positron the overlapping arms* and *(ii) electron in the overlapping, positron the non-overlapping arms* take place at the same time, had not the weak value of the third pair occupation operator $\bar{N}(1 - |1+) = 1 - \bar{N}(1 - |2+) - \bar{N}(2 - |1+)$ turned out to be -1 as required by Eqs. (31) and (32). The authors of Ref. [5] see in this "remarkable way" in which quantum mechanics solves the paradox of having more than one electron-positron pair involved.

Here we adopt a different view. Clearly, $\bar{N}(1 - |1+) = -1$ is not a valid pair occupational number for the non-overlapping arms of the interferometers, and as long as it is an essential part of the reasoning, we rather doubt the whole "resolution" of the above paradox. It is easy to see how this anomalous value arises. Measurement of $\bar{N}(1 - |1+) = diag(1,0,0,0)$ creates two pathways, $\{1\}$ with the probability amplitude $1/4$, labelled 1, and $\{2+3\}$ with the amplitude $-1/2$, labelled 0 in Fig. 4. In the weak limit (29), the two pathways interfere and we have, in fact, a double-slit experiment, where we try to determine the chosen slit without destroying the interference pattern on the screen, which the uncertainty principle forbids [9, 10]. A weak meter complies with the uncertainty principle by yielding a value which may lie anywhere on the real axis, and not just between 0 and 1. The mathematical reason for this is that the averaging in the r.h.s. of Eq. (30) is done with an improper alternating distribution [16], and the negative occupation number just manifests the failure to sensibly answer the "which way?" question.

This can be illustrated further by considering a more extreme case with the final state (12) replaced by (we assume that we can do that)

$$|f\rangle = (|1-\rangle|1+\rangle - |1-\rangle|2+\rangle - \varepsilon|2-\rangle|1-\rangle + \varepsilon|2-\rangle|2+\rangle)/2^{1/2}(1+\varepsilon)^{1/2}, \quad (35)$$

where ε is a parameter, so that for $\varepsilon = 1$ we recover the original transition in the Hardy's set up. Note that $\hat{N}(2 - |1+)$ creates a single pathway $\{3\}$, since the amplitude for $\{1+2\}$ vanishes, whereas for $\hat{N}(1 - |1+)$

9

and $\hat{N}(1-|2+)$ there exist two pathways with non-zero amplitudes. The weak pair occupation numbers in (30) now are

$$
\begin{aligned}
\bar{N}(1-|1+) &= -1/\varepsilon \\
\bar{N}(1-|2+) &= 1/\varepsilon \\
\bar{N}(2-|1+) &= 1.
\end{aligned}
\tag{36}
$$

Similarly, for the single particle occupation numbers we find

$$
\begin{aligned}
\bar{N}(1+) &= 1-1/\varepsilon \\
\bar{N}(2-) &= 1.
\end{aligned}
\tag{37}
$$

Now let $\varepsilon \to 0$ so that the transition becomes very improbable. For $\varepsilon = 10^{-6}$ the last of Eqs. (36) suggests that, in defiance of the uncertainty principle, we have established that in a system not perturbed by observations electron and positron always travel along the overlapping and non-overlapping arms, respectively. One must, however, add that there are also a million electron-positron pairs travelling along the non-overlapping and overlapping arms, and also minus one million pairs in the non-overlapping arms. We might try to retain only the results where a weak value is obtained under the non-interference condition (33) but then we would have to discard also the value $\bar{N}(1-|1+) = -1$ needed to balance the books in the original example of Ref. [5]. A more realistic view [16] is that the unusual properties of the weak values signal a failure of our measurement procedure in the case when it is expected to fail. This failure occurs in a consistent way and can easily be observed, e.g., in an optical experiment [15]. What one observes, however, are the properties of a meter in a regime where it ceases to be a proper meter, not to be confused with the attributes of the measured system which, quantum mechanics tells us, simply do not exist. For example, treating $\bar{N}(1+)$ as the actual charge in the non-overlapping positron arm [5] would mean that with no effort whatsoever (although not very often) we may create an arbitrary large charge, perhaps exceeding the largest one allowed by the relativistic quantum mechanics [26].

8. Conclusions and discussion

In summary, Feynman path approach offers a convenient language for describing some controversial aspects of quantum measurement theory. A quantum system can be seen as arriving in a given final state via a set of virtual (Feynman) paths defined for a particular "position operator" $\hat{n}$, all of whose eigenvalues are non-degenerate. With no other measurements performed, the paths form one indivisible (real) pathway. Intermediate von Neumann measurements of operators which commute with $\hat{n}$ produce, by combining Feynman paths into classes and destroying coherence between them, a network of real pathways connecting the initial and final states of the system. The probabilities which can now be assigned to the pathways define a classical statistical ensemble which one observes in an experiment. The following statement can be regarded as a corollary to the Feynman's uncertainty principle [9]: each set of intermediate measurements produces a different network of real pathways, in such a way that in general no properties of a network A can be inferred from a network B, and, similarly, no properties of the unobserved system can be inferred from either of the networks. It is readily seen that an attempt to ascribe properties of different networks (e.g., the four italicised statements of Section 5) to a single (e.g., unobserved) system may lead to contradiction. Such attempts constitute counterfactual reasoning which, as has been noticed before [3, 4] ought to be avoided. For the Hardy's set up we find, for example, that

(*I*) electron and positron always travel along the non-overlapping and overlapping arms, respectively, (path {2} in (10)) *if {2} is decohered, while coherence between Feynman paths {1} and {3} is not destroyed*,

(*II*) electron and positron always travel along the overlapping and non-overlapping arms, respectively, (path {3} in (10)) *if {3} is decohered, while coherence between Feynman paths {1} and {2} is not destroyed*,

which presents no contradiction with the qualifications added.

We argue further that weak measurements do not provide a sufficient justification for counterfactual reasoning. It is important to go through the statements made in [5] in greater detail. It is true that where only one path contributes to a transition, a weak value would coincide with that obtained in an ideal strong measurement - Eq. (5) shows that that is the case for a meter of arbitrary accuracy. One cannot, however, consistently avoid the "strange" anomalous weak values where there is more than one interfering paths (cf. the exceptionally large occupation numbers discussed in Section 6). Reference [5] tells us that "strangeness is not a problem; consistency is the real issue". This is not quite so: one is lead to believe that a weak measurement is, in some sense, a valid extension of the accurate von Neumann measurement. This notion largely accounts for one's interest in the subject as well as for one's surprise when a "strange" reading, such as a slit number 10 in the two-slit case, is produced. An analogy with a purely classical meter employed to measure the slit number for a classical particle is helpful: the meter malfunctions and produces readings of, say, 10 if the slit 1 is used and 15 if the slit 2 is used. The link between a reading and the measured attribute of the system (slit number 1 or 2) is lost and, unless one knows how to re-calibrate the meter, the measurement becomes a study of the meter's rather than the system's properties. The peculiarity of the quantum situation is that a correct suitable answer to the question just does not exist under the weak conditions and a weak meter cannot be re-calibrated. In consequence, a weak measurement ceases to be a valid extension of the von Neumann procedure, and its result becomes a manifestation of the failure to find a suitable value of the system's intrinsic attribute which, quantum mechanics tells us, is unavailable. Accordingly, the meter, which can no longer be a good one, behaves *as if* the slit number were 10 or *as if* there were 10^6 electron-positron pairs in a situation where it can be verified independently that only two holes have been drilled in the screen and only two particles were injected into the system. Many authors [5, 13, 15] have emphasised the "surprising" aspect of such anomalous weak values. It would, however, been far more surprising had weak measurements consistently produced "suitable" results consistent, for example, with unobserved particle passing through each slit with a certain probability. Existence of strange anomalous values is a proof of the contrary and can be seen as a direct consequence of the uncertainty principle [16]. That not all weak values are strange and that the strange ones usually occur where there is destructive interference between virtual paths does not alter this conclusion. Finally, it is not surprising that a weak meter and the weak values obey their own self-consistent logic [5]. This would be the case for any degree of freedom whose interaction with another system is described by a reasonable coupling. This logic does not, however, extend to (non-existing) intrinsic properties of the measured system, as is required if one wishes to make an argument in favor of counterfactual reasoning. For this reason, weak measurements do not extend the limit set by the uncertainty principle on what can be learned about a quantum system. It seems appropriate to conclude with Feynman's own warning against such an extension [27]: "Do not keep saying to yourself, if you can possibly avoid it, "But how can it be like that?" because you will get "down the drain", into a blind alley from which nobody has yet escaped."

Acknowledgemnts

References

[1] L. Hardy, Phys. Rev. Lett. **68** (1992) 2981.

[2] L. Vaidman, Phys. Rev. Lett. **70** (1993) 3369.

[3] H. S. Stapp Am. J. Phys. **65** (1997) 300; Am. J. Phys. **66** (1998) 924.

[4] W. Unruh, Phys. Rev. A **59** (1997) 126.

[5] Y. Aharonov, A. Botero, S. Popescu, B. Reznik and J. Tollaksen, Phys. Lett. A **301** (2002) 130.

[6] S. E. Ahnert and M. C. Payne, Phys. Rev. A **70** (2004) 042102.

[7] J. S. Lundeen and K. J. Resch, Phys. Lett. A **334** (2005) 337.

[8] J. S. Lundeen, K. J. Resch and A. M. Steinberg, Phys. Rev. A **72** (2005) 016101.

[9] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, New York 1965).

[10] R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman lectures on physics: quantum mechanics*, (Addison-Wesley, 1965).

[11] Y. Aharonov, D. Z. Albert and L. Vaidman, Phys. Rev. Lett. **60** (1988) 1351.

[12] Y. Aharonov and L. Vaidman, Phys. Rev. A **41** (1990) 11.

[13] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics*, edited by J. G. Muga, R. Sala Mayato and I. L. Egusquiza (Springer, 2002), pp. 369-413.

[14] Y. Aharonov and L. Vaidman, J. Phys. A **24** (1991) 2315.

[15] N. W. M. Ritchie, J. G. Story and R. G. Hulet, Phys. Rev. Lett. **66** (1991) 1107.

[16] D. Sokolovski, Phys. Rev. A **76** (2007) 042125.

[17] The original paper on weak measureaments was [11] entitled "How can a measurement of a spin 1/2 give a result 100?". While conceptually identical, the double-slit (Mach-Zehnder interferometer) example provides somewhat stronger case against over-interpretation of weak values. Unlike the value of a spin component, the number of slits (arms) can established independently by purely classical methods. Thus any measurement result which appears to imply the existence of more than two slits or arms would require an explanation of the apparent contradiction.

[18] Compare with the position operator for a particle in one spacial dimension, $\hat{x} = \int |x\rangle x \langle x| dx$.

[19] D. Sokolovski and R. Sala Mayato, Phys.Rev. A **71** (2005) 042101.

[20] D. Sokolovski and R. Sala Mayato, Phys.Rev. A **73** (2006) 052115; **76** (2007) 039903(E).

[21] T. Ravon and L. Vaidman, J. Phys. A **40** (2007) 2873.

[22] C. R. Leavens, I. Puerto Giménez, D. Alonso and R. Sala Mayato, Phys. Lett. A **359** (2006) 416.

[23] D. Sokolovski, I. Puerto Giménez and R. Sala Mayato (unpublished).

[24] C. Cohen-Tanoudji, B. Diu and F. Laloë, *Quantum Mechanics, Volume one*, (Hermann and John Wiley & Sons., 1977)

[25] J. Bub and H. Brown, Phys. Rev. Lett. **56** (1986) 2337.

[26] E. M. Lifshitz, V. B. Berestetskii and L. P. Pitaevski, *Quantum Electrodynamics*, (Reed Elsevier, 1982)

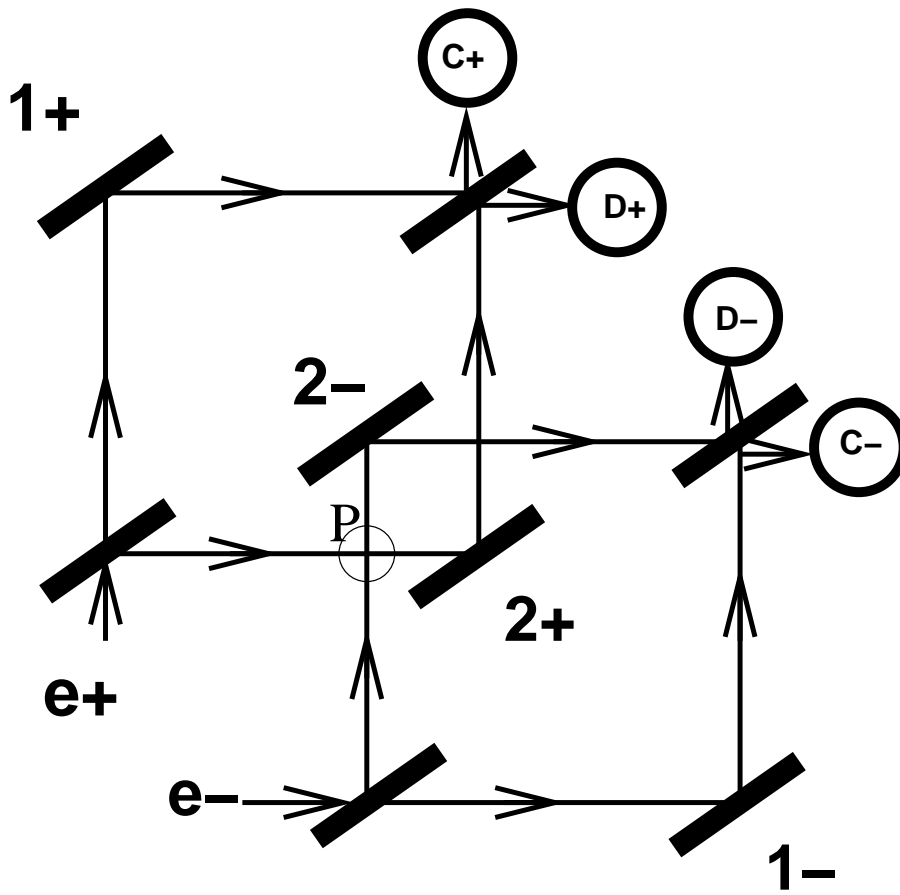[27] R. P. Feynman, *The character of physical law*, (Modern Library, 1994).

Figure 1: The Hardy's set up of mirrors and beam splitters. An electron ($e-$) and a positron ($e+$) are simultaneously injected into the system. Their simultaneous presence in the overlapping arms $2-$ and $2+$, respectively, leads to certain annihilation.
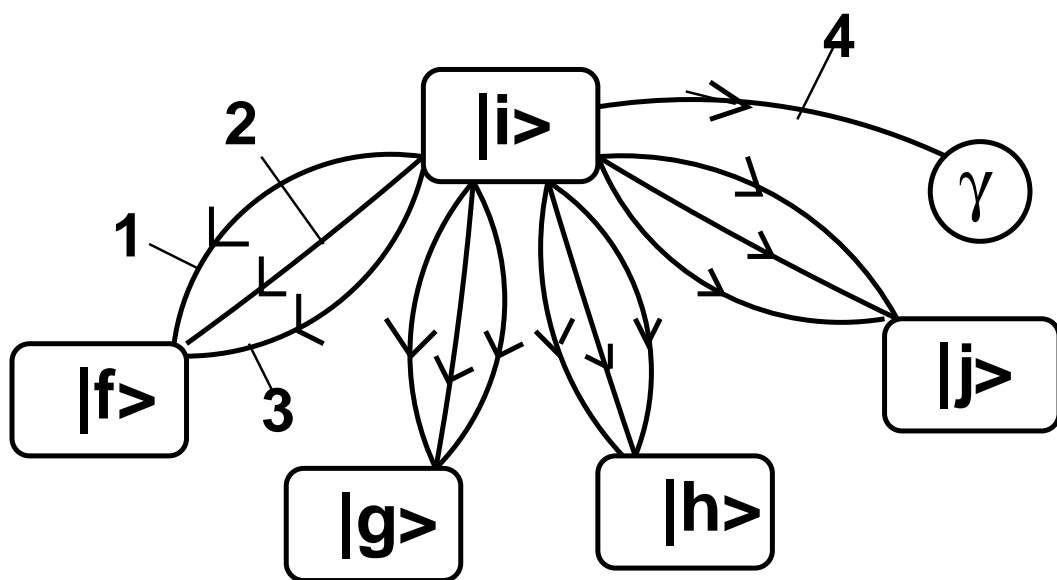
Figure 2: Five final states and twelve paths connecting them with the initial state $|i\rangle$.

Path	$\Phi_{\{n\}}^{f \leftarrow i}$	$\Phi_{\{n\}}^{g \leftarrow i}$	$\Phi_{\{n\}}^{h \leftarrow i}$	$\Phi_{\{n\}}^{j \leftarrow i}$
{1}	1/4	1/4	1/4	1/4
{2}	-1/4	1/4	-1/4	1/4
{3}	-1/4	-1/4	1/4	1/4
{4}	0	0	0	0

Figure 3: Table 1. Probability amplitudes for the virtual paths shown in Fig. 2.
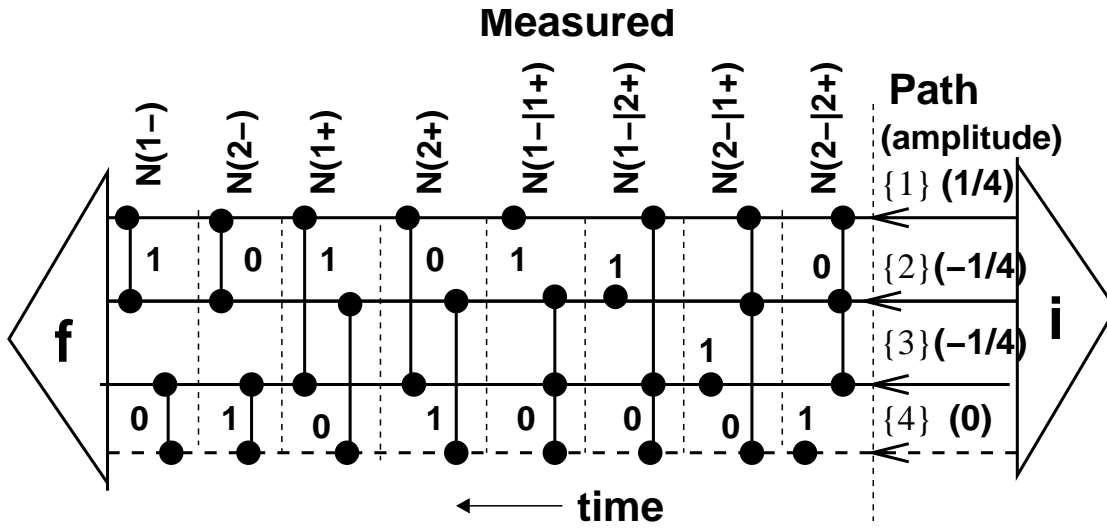
Figure 4: Virtual paths in (10) which connect the states $|i\rangle$ and $|f\rangle$. The path $\{4\}$ has a zero amplitude due to annihilation, and is shown by a dashed line. An accurate measurement of one of the operators listed above creates two real pathways, each comprising the paths joined by the vertical lines. The nunmbers 1 or 0 next to the lines are the eigenvalues of the measured operator.

Measured	Paths	$P^{f \leftarrow i}$	$P^{g \leftarrow i}$	$P^{h \leftarrow i}$	$P^{j \leftarrow i}$
Nothing	{1+2+3}	1/16	1/16	1/16	9/16
N(1-|1+)	{1}	1/16			
	{2+3}	1/4	5/16	1/16	1/16	5/16
N(1-|2+)	{2}	1/16			
	{1+3}	0	1/16	1/16	5/16	5/16
N(2-|1+)	{3}	1/16			
	{1+2}	0	1/16	5/16	1/16	5/16
N(1-)	{1+2}	0			
	{3}	1/16	1/16	5/16	1/16	5/16
N(2-)	{3}	1/16			
	{1+2}	0	1/16	5/16	1/16	5/16
N(1+)	{1+3}	0			
	{2}	1/16	1/16	1/16	5/16	5/16
N(2+)	{2}	1/16			
	{1+3}	0	1/16	1/16	5/16	5/16

Figure 5: Table 2. Transition probabilities for the final states in Fig.2 if intermediate measurements in Fig.4 are performed. Also shown are the probabilities for the real pathways created by measurements for the final state $|f\rangle$ in (12). In all cases the annihilation occurs with probability of 1/4.